GENERALIZED COHOMOLOGY
FOR IRREDUCIBLE TENSOR FIELDS
OF MIXED YOUNG SYMMETRY TYPE

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Abstract

We construct N-complexes of non completely antisymmetric irreducible tensor fields on $\mathbb{R}^D$ generalizing thereby the usual complex ($N = 2$) of differential forms. These complexes arise naturally in the description of higher spin gauge fields. Although, for $N \geq 3$, the generalized cohomology of these N-complexes is non trivial, we prove a generalization of the Poincaré lemma. Several results which appeared in various contexts are shown to be particular cases of this generalized Poincaré lemma.

1 Introduction

Our aim in this letter is to set up differential tools for irreducible tensor fields on $\mathbb{R}^D$ which generalize the calculus of differential forms. By an irreducible tensor field on $\mathbb{R}^D$, we here mean, a smooth mapping $x \mapsto T(x)$ of $\mathbb{R}^D$ into a vector space of (covariant) tensors of given Young symmetry. We recall that this implies that the representation of GL_D in the corresponding space of tensors is irreducible.

We first introduce a generalization of the familiar exterior derivative that satisfies, instead of $d^2 = 0$, the nilpotency condition $d^N = 0$ for some integer $N \geq 2$ that depends on the Young symmetry type of the tensor fields under consideration. We then analyse the generalized (co)homologies $H^{(k)} \equiv Ker d^k / Im d^{N-k}$, $(k = 1, \cdots, N - 1)$ of these nilpotent endomorphisms in the sense of [3], [4], [5], [7], [10], [11] and establish an analog of the Poincaré lemma.

The nilpotent endomorphisms introduced here have various physical applications. They naturally arise, for instance, in the theory of higher spin gauge fields. They also encompass conservation laws involving symmetric tensors. This is discussed at the end of the letter.

An expanded version of this letter, with further developments and detailed proofs, will appear elsewhere [6].

2 Definitions

Throughout the following $(x^\mu) = (x^1, \ldots, x^D)$ denotes the canonical coordinates of $\mathbb{R}^D$ and ∂_μ are the corresponding partial derivatives which we identify with the corresponding covariant derivatives associated to the canonical flat linear connection of $\mathbb{R}^D$. Thus, for instance, if T is a covariant tensor field of degree p on $\mathbb{R}^D$ with components $T_{\mu_1 \ldots \mu_p}(x)$, then ∂T denotes the covariant tensor field of degree $p + 1$ with components $\partial_{\mu_1} T_{\mu_2 \ldots \mu_{p+1}}(x)$. The operator ∂ is a first-order differential operator which increases by one the tensorial degree.

In this context, the space $\Omega(\mathbb{R}^D)$ of differential forms on $\mathbb{R}^D$ is the graded vector space of (covariant) antisymmetric tensor fields on $\mathbb{R}^D$ with graduation induced by the tensorial degree whereas the exterior differential d is the compo-

sition of the above ∂ with antisymmetrisation, i.e.

$$d = \mathbf{A}_{p+1} \circ \partial : \Omega^p(\mathbb{R}^D) \to \Omega^{p+1}(\mathbb{R}^D) \tag{1}$$

where $\mathbf{A}_p$ denotes the antisymmetrizer on tensors of degree p. The Poincaré lemma asserts that the cohomology of the complex $(\Omega(\mathbb{R}^D), d)$ is trivial, i.e. that one has $H^p(\Omega(\mathbb{R}^D)) = \mathrm{Ker}(d : \Omega^p(\mathbb{R}^D) \to \Omega^{p+1}(\mathbb{R}^D))/d(\Omega^{p-1}(\mathbb{R}^D)) = 0$, $\forall p \geq 1$ and $H^0(\Omega(\mathbb{R}^D)) = \mathrm{Ker}(d : \Omega^0(\mathbb{R}^D) \to \Omega^1(\mathbb{R}^D)) = \mathbb{R}$.

From the point of view of Young symmetry, antisymmetric tensors correspond to Young diagrams (partitions) described by one column of cells, i.e. the space of values of p-forms corresponds to one column of p cells, (1^p), whereas $\mathbf{A}_p$ is the associated Young symmetrizer.

There is a relatively easy way to generalize the pair $(\Omega(\mathbb{R}^D), d)$ which we now describe. Let $Y = (Y_p)_{p \in \mathbb{N}}$ be a sequence of Young diagrams such that the number of cells of Y_p is p, $\forall p \in \mathbb{N}$ (i.e. such that Y_p is a partition of the integer p for any p). We define $\Omega_Y^p(\mathbb{R}^D)$ to be the vector space of smooth covariant tensor fields of degree p on $\mathbb{R}^D$ which have the Young symmetry type Y_p and we let $\Omega_Y(\mathbb{R}^D)$ be the graded vector space $\oplus_p \Omega_Y^p(\mathbb{R}^D)$. We then generalize the exterior differential by setting $d = \mathbf{Y} \circ \partial$, i.e.

$$d = \mathbf{Y}_{p+1} \circ \partial : \Omega_Y^p(\mathbb{R}^D) \to \Omega_Y^{p+1}(\mathbb{R}^D) \tag{2}$$

where $\mathbf{Y}_p$ is now the Young symmetrizer on tensor of degree p associated to the Young symmetry Y_p. This d is again a first order differential operator which is of degree one, (i.e. it increases the tensorial degree by one), but now, $d^2 \neq 0$ in general. Instead, one has the following result.

LEMMA 1 *Let N be an integer with $N \geq 2$ and assume that Y is such that the number of columns of the Young diagram Y_p is strictly smaller than N (i.e. $\leq N - 1$) for any $p \in \mathbb{N}$. Then one has $d^N = 0$.*

In fact the indices in one column are antisymmetrized and $d^N \omega$ involves necessarily at least two partial derivatives ∂ in one of the columns since there are N partial derivatives involved and at most $N - 1$ columns.

Thus if Y satisfies the condition of Lemma 1, $(\Omega_Y(\mathbb{R}^D), d)$ is a N-*complex* (of cochains) [10], [3], [7], [11], [4], i.e. here a graded vector space equipped with an endomorphism d of degree 1, its N-*differential*, satisfying $d^N = 0$. Concerning N-complexes, we shall use here the notations and the results [4].

Notice that $\Omega_Y^p(\mathbb{R}^D) = 0$ if the first column of Y_p contains more than D cells and that therefore, if Y satisfies the condition of Lemma 1, then $\Omega_Y^p(\mathbb{R}^D) = 0$ for $p > (N-1)D$.

One can also define a graded bilinear product on $\Omega_Y(\mathbb{R}^D)$ by setting

$$(\alpha\beta)(x) = \mathbf{Y}_{a+b}(\alpha(x) \otimes \beta(x)) \tag{3}$$

for $\alpha \in \Omega_Y^a(\mathbb{R}^D)$, $\beta \in \Omega_Y^b(\mathbb{R}^D)$ and $x \in \mathbb{R}^D$. This product is by construction bilinear with respect to the $C^\infty(\mathbb{R}^D)$-module structure of $\Omega_Y(\mathbb{R}^D)$ (i.e. with respect to multiplication by smooth functions). It is worth noticing here that one always has $\Omega_Y^0(\mathbb{R}^D) = C^\infty(\mathbb{R}^D)$.

3 The N-complexes $(\Omega_N(\mathbb{R}^D), d)$

In this letter, we shall not stay at this level of generality but, for each $N \geq 2$ we shall choose a maximal Y, denoted by $Y^N = (Y_p^N)_{p \in \mathbb{N}}$, satisfying the condition of lemma 1. The Young diagram with p cells Y_p^N is defined in the following manner: write the division of p by $N-1$, i.e. write $p = (N-1)n_p + r_p$ where n_p and r_p are (the unique) integers with $0 \leq n_p$ and $0 \leq r_p \leq N-2$ (n_p is the quotient whereas r_p is the remainder), and let Y_p^N be the Young diagram with n_p rows of $N-1$ cells and the last row with r_p cells (if $r_p \neq 0$). One has $Y_p^N = ((N-1)^{n_p}, r_p)$, that is we fill the rows maximally. We shall denote $\Omega_{Y^N}(\mathbb{R}^D)$ and $\Omega_{Y^N}^p(\mathbb{R}^D)$ by $\Omega_N(\mathbb{R}^D)$ and $\Omega_N^p(\mathbb{R}^D)$. It is clear that $(\Omega_2(\mathbb{R}^D), d)$ is the usual complex $(\Omega(\mathbb{R}^D), d)$ of differential forms on $\mathbb{R}^D$. The N-complex $(\Omega_N(\mathbb{R}^D), d)$ will be simply denoted by $\Omega_N(\mathbb{R}^D)$.

We call the Young diagrams Y_p^N with $p = (N-1)n_p$ "well-filled diagrams". These are rectangular diagrams with n_p rows of $N-1$ cells each.

We recall [4] that the (generalized) cohomology of the N-complex $\Omega_N(\mathbb{R}^D)$ is the family of graded vector spaces $H_{(k)}(\Omega_N(\mathbb{R}^D))$ $k \in \{1, \ldots, N-1\}$ defined by

$$H_{(k)}(\Omega_N(\mathbb{R}^D)) = \mathrm{Ker}(d^k)/\mathrm{Im}(d^{N-k}), \text{ i.e. } H_{(k)}(\Omega_N(\mathbb{R}^D)) = \bigoplus_p H_{(k)}^p(\Omega_N(\mathbb{R}^D)) \text{ with}$$

$$H_{(k)}^p(\Omega_N(\mathbb{R}^D)) = \mathrm{Ker}(d^k : \Omega_N^p(\mathbb{R}^D) \to \Omega_N^{p+k}(\mathbb{R}^D))/d^{N-k}(\Omega^{p+k-N}(\mathbb{R}^D)).$$

It is easy to write down explicit formulas in terms of components. Consider for instance the case $N = 3$, for which the relevant Young diagrams are those with two colums, one of length k and the second of length $k-1$ or k. A tensor field in $\Omega_3(\mathbb{R}^D)$ is a scalar T in tensor degree 0, a vector T_α in tensor degree 1, a symmetric tensor $T_{\alpha\beta}$ in tensor degree 2. In tensor degree $2k - 1$ ($k \geq 2$), it is described by components $T_{\alpha_1\ldots\alpha_k\beta_1\ldots\beta_{k-1}}$ with the Young symmetry of the diagram with $k-1$ rows of length 2 and one row of length 1, while in even tensor degree $2k$, it is described by components $T_{\alpha_1\ldots\alpha_k\beta_1\ldots\beta_k}$ with the Young symmetry of the well-filled rectangular diagram with k rows of length 2. The components of dT are respectively proportional to $\partial_\alpha T$, $\partial_{(\alpha} T_{\beta)}$, $\partial_{[\alpha_1} T_{\alpha_2]\beta}$ and $T_{\alpha_1\ldots\alpha_k[\beta_2\ldots\beta_k,\beta_1]} + T_{\beta_1\ldots\beta_k[\alpha_2\ldots\alpha_k,\alpha_1]}$ or $\partial_{[\alpha_1} T_{\alpha_2\ldots\alpha_{k+1}]\beta_1\ldots\beta_k}$, where the comma stands for the partial derivative, $(\ldots)$ for symmetrization and $[\ldots]$ for antisymmetrization. It is obvious that $d^3 = 0$ since all terms in $d^3 T$ involves one antisymmetrization over partial derivatives.

4 Generalized Poincaré lemma

The following statement is our generalization of the Poincaré lemma.

THEOREM 1 *One has $H_{(k)}^{(N-1)n}(\Omega_N(\mathbb{R}^D)) = 0$, $\forall n \geq 1$ and $H_{(k)}^0(\Omega_N(\mathbb{R}^D))$ is the space of real polynomial functions on $\mathbb{R}^D$ of degree strictly less than k (i.e. $\leq k - 1$) for $k \in \{1, \ldots, N - 1\}$.*

This statement reduces to the Poincaré lemma for $N = 2$ but it is a nontrivial generalization for $N \geq 3$ in the sense that the spaces $H_{(k)}^p(\Omega_N(\mathbb{R}^D))$ are nontrivial for $p \neq (N - 1)n$ and, in fact, are generically infinite dimensional for $D \geq 3$, $p \geq N$.

The second part of the theorem is obvious since the condition $d^k f = 0$ simply states that the derivatives of order k of f all vanish (and there is no quotient to be taken since f is in degree 0). The proof of the first part of the theorem, which asserts that there is no cohomology for well-filled diagrams, proceeds by

introducing an appropriate generalized homotopy [4]. By inner contraction with the vector x^μ, one can easily construct from a well-filled tensor field $R^{(N-1)n}$ of degree $(N-1)n$ with $n \geq 1$ fulfilling $d^k R^{(N-1)n} = 0$, a tensor field $K^{(N-1)(n-1)+k-1}$ (of degree $(N-1)(n-1)+k-1$) such that $R^{(N-1)n} = d^{N-k}K^{(N-1)(n-1)+k-1}$. The construction works only for well-filled tensors; for tensors of a different Young symmetry type, the tensor K obtained through the homotopy in the given N-complex does not fulfill $d^{N-k}K = R$, (for $d^k R = 0$).

The details will be given in [6]. We shall merely display here two explicit homotopy formulas which reveal the main points and which deals with cohomologies effectively investigated in the literature previously (see next section). Consider first in $\Omega_4(\mathbb{R}^D)$ a tensor T in degree 3 which is annihilated by d^3. In components,

$$\partial_{[\alpha_1}\partial_{[\beta_1}\partial_{[\gamma_1}T_{\alpha_2]\beta_2]\gamma_2]} = 0$$

where the antisymmetries are on the α's, the β's and the γ's. A straightforward calculation shows that $d^3 T = 0$ implies $T = d\xi$ ($\leftrightarrow T_{\alpha\beta\gamma} = \partial_{(\alpha}\xi_{\beta\gamma)}$), with $\xi_{\alpha\beta}$ given by the homotopy formula

$$
\begin{aligned}
\xi_{\alpha\beta}(x) \;=\; & \int_0^1 dt\, T_{\alpha\beta\lambda}(tx)\, x^\lambda \\
& + \frac{1}{2} \int_0^1 dt \int_0^t dt'\, (\partial_{[\mu}T_{\alpha]\beta\lambda}(t'x) + \partial_{[\mu}T_{\beta]\alpha\lambda}(t'x))\, x^\mu\, x^\lambda \\
& + \int_0^1 dt \int_0^t dt' \int_0^{t'} dt''\, \partial_{[\mu}\partial_{[\rho}T_{\alpha]\beta]\lambda}(t''x)\, x^\lambda\, x^\mu\, x^\rho.
\end{aligned}
\tag{4}
$$

Thus, $H^3_{(3)}(\Omega_4(\mathbb{R}^D)) = 0$. In the homotopy formula (4), not only does the inner contraction of T with x appear, but also the double contraction of dT with xx, as well as the triple contraction of d^2T with xxx.

The second illustrative homotopy formula shows that $H^4_{(1)}(\Omega_3(\mathbb{R}^D)) = 0$. If the tensor $R_{\alpha_1\alpha_2\beta_1\beta_2}$ of degree 4 with the symmetry

α_1	β_1
α_2	β_2

, (i.e. the symmetry of Riemann curvature tensor) is annihilated by d, $\partial_{[\alpha_3}R_{\alpha_1\alpha_2]\beta_1\beta_2} = 0$, then one has $R_{\alpha_1\alpha_2\beta_1\beta_2} = 4\partial_{[\beta_2}\partial_{[\alpha_2}h_{\alpha_1]\beta_1]}$ with

$$h_{\alpha\beta}(x) = \int_0^1 dt \int_0^t dt'\, t'\, x^\lambda x^\mu R_{\alpha\lambda\beta\mu}(t'x). \tag{5}$$

Both homotopy formulas given here follow the general pattern described in [4]. The general case for arbitrary N, n or k in the theorem leads to homotopy formulas with the same structure.

One can alternatively prove the theorem by repeated use of the standard Poincaré lemma for differential forms, but this appears to be more laborious for big N.

Although there is no cohomology for well-filled tensors, the cohomology is non trivial at the other tensorial degrees. One easily verifies that the cohomology for tensors corresponding to a single (unfilled) row is finite-dimensional and related to the Killing tensors of Minkowski space. The cohomology in the other cases, however, is generically infinite-dimensional. One may remove it by embedding the N-complex $(\Omega_N(\mathbb{R}^D), d)$ in a bigger N-complex, containing different symmetry types (and thus reducible tensors) in each tensorial degree, but this will not be done here. Again, the details will be given in [6].

5 Higher spin gauge fields

The N-complexes (and their generalized cohomologies) defined in this letter naturally arise in the description of higher integer spin gauge fields.

Classical spin S gauge fields (with $S \in \mathbb{N}$) are described by symmetric tensor fields $h_{\alpha_1 \ldots \alpha_S}$ of order S and gauge transformations of the form

$$\delta_\epsilon h_{\alpha_1 \ldots \alpha_S} = \partial_{(\alpha_1} \epsilon_{\alpha_2 \ldots \alpha_S)} \tag{6}$$

where $\epsilon_{\alpha_2 \ldots \alpha_S}$ is a symmetric tensor of order $S-1$. [3]. The curvatures $R_{\alpha_1 \ldots \alpha_S \beta_1 \ldots \beta_S}$ invariant under (6) contain S derivatives of the fields [1] and are obtained from $\partial_{\alpha_1 \ldots \alpha_S} h_{\beta_1 \ldots \beta_S}$ by symmetrizing according to the Young tableau with S columns and 2 rows.

It is clear from the above definitions that $R = d^S h$ where d is the derivative operator of the complex $(\Omega_{S+1}(\mathbb{R}^D), d)$. Gauge invariance of the curvature follows

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[3]For $S \geq 3$, the gauge parameter is subject to the trace condition $\epsilon^{\alpha_2}{}_{\alpha_2 \ldots \alpha_{S_1}} = 0$ and for $S \geq 4$, the gauge field is subject to the double-trace condition $h^{\alpha_1 \alpha_2}{}_{\alpha_1 \alpha_2 \ldots \alpha_{S-2}} = 0$ [8, 12]. However, as observed in [1, 2], it is already of interest to investigate the gauge symmetries without imposing the trace conditions.

from $d^{S+1} = 0$.

The generalized Poincaré lemma (Theorem 1) implies $H^S_{(S)}(\Omega_{S+1}(\mathbb{R}^D)) = 0$ which ensures that gauge fields with zero curvatures are pure gauge. This was directly proved in [2] for the case $S = 3$. The condition $d^{S+1} = 0$ also ensures that curvatures of gauge potentials satisfy a generalized Bianchi identity of the form $dR = 0$. The generalized Poincaré lemma also implies $H^{2S}_{(1)}(\Omega_{S+1}(\mathbb{R}^D)) = 0$ which means that conversely the Bianchi identity characterizes the elements of $\Omega^{2S}(\mathbb{R}^D)$ which are curvatures of gauge potentials. This claim for $S = 2$ is the main statement of [9].

6   Duality

Finally, there is a generalization of Hodge duality for $\Omega_N(\mathbb{R}^D)$, which is obtained by contractions of the columns with the Kroneker tensor $\varepsilon^{\mu_1 \cdots \mu_D}$ of $\mathbb{R}^D$. A detailed description of this duality will appear in [6]. When combined with Theorem 1, this duality leads to another kind of results. A typical result of this kind is the following one. Let $T^{\mu\nu}$ be a symmetric contravariant tensor field of degree 2 on $\mathbb{R}^D$ satisfying $\partial_\mu T^{\mu\nu} = 0$, (like e.g. the stress energy tensor), then there is a contravariant tensor field $R^{\lambda\mu\rho\nu}$ of degree 4 with the symmetry $\begin{array}{|c|c|} \hline \lambda & \rho \\ \hline \mu & \nu \\ \hline \end{array}$, (i.e. the symmetry of Riemann curvature tensor), such that

$$T^{\mu\nu} = \partial_\lambda \partial_\rho R^{\lambda\mu\rho\nu} \tag{7}$$

In order to connect this result with Theorem 1, define $\tau_{\mu_1\ldots\mu_{D-1}\nu_1\ldots\nu_{D-1}} = T^{\mu\nu}\varepsilon_{\mu\mu_1\ldots\mu_{D-1}}\varepsilon_{\nu\nu_1\ldots\nu_{D-1}}$. Then one has $\tau \in \Omega^{2(D-1)}_3(\mathbb{R}^D)$ and conversely, any $\tau \in \Omega^{2(D-1)}_3(\mathbb{R}^D)$ can be expressed in this form in terms of a symmetric contravariant 2-tensor. It is easy to verify that $d\tau = 0$ (in $\Omega_3(\mathbb{R}^D)$) is equivalent to $\partial_\mu T^{\mu\nu} = 0$. On the other hand, Theorem 1 implies that $H^{2(D-1)}_{(1)}(\Omega_3(\mathbb{R}^D)) = 0$ and therefore $\partial_\mu T^{\mu\nu} = 0$ implies that there is a $\rho \in \Omega^{2(D-2)}_3(\mathbb{R}^D)$ such that $\tau = d^2\rho$. The latter is equivalent to (7) with $R^{\mu_1\mu_2}{}_{\nu_1\nu_2}$ proportional to $\varepsilon^{\mu_1\mu_2\ldots\mu_D}\varepsilon^{\nu_1\nu_2\ldots\nu_D}\rho_{\mu_3\ldots\mu_D\nu_3\ldots\nu_D}$ and one verifies that, so defined, R has the correct symmetry. That symmetric tensor fields identically fulfilling $\partial_\mu T^{\mu\nu} = 0$ can be rewritten as in Eq. (7) has

been used in [13] in the investigation of the consistent deformations of the free spin two gauge field action.

7 The differential calculus for a manifold

If the space $\mathbb{R}^D$ is replaced by an arbitrary D-dimensional smooth manifold V, then smooth covariant tensor fields of given Young symmetry type are still well defined and therefore the graded space $\Omega_Y(V) = \oplus_p \Omega_Y^p(V)$ is well defined for a sequence $Y = (Y_p)_{p \in \mathbb{N}}$ of Young diagrams such that Y_p has p cells $\forall p \in \mathbb{N}$ as in Section 2. In fact $\Omega_Y(V)$ is a graded module over the algebra $C^\infty(V)$ of smooth functions and (3) still defines a $C^\infty(V)$-bilinear graded product on $\Omega_Y(V)$. However now the operator $T \mapsto \partial T$ of Section 2 does not make sense; in order to give a substitute for it, one must choose a linear connection on V and replace ∂ by the corresponding covariant derivative ∇. One then generalizes d by $d_\nabla = \mathbf{Y} \circ \nabla$, i.e. formula (2) by

$$d_\nabla = \mathbf{Y}_{p+1} \circ \nabla : \Omega_Y^p(V) \to \Omega_Y^{p+1}(V) \tag{8}$$

which defines again a first order differential operator on $\Omega_Y(V)$. This operator d_∇ is again homogeneous of degree 1 but now, due to the torsion and the curvature of ∇, Lemma 1 is not true. In fact Lemma 1 merely applies at the level of symbols; more precisely one has the following: *Let N and Y satisfy the assumptions of Lemma 1, then $(d_\nabla)^N$ is a differential operator of order smaller or equal to $N-1$ and, if furthermore ∇ is torsion-free, then the order of $(d_\nabla)^N$ is smaller or equal to $N-2$.* In the case $N = 2$, if ∇ is torsion free, $(d_\nabla)^2 = 0$ follows from the first Bianchi identity; however in this case d_∇ coincides, as well known, with the exterior differential d which is well defined in local coordinates by (1).

Acknowledgements

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